ON THE IMPLEMENTATION OF 2-BAND CYCLIC FILTERBANKS

M.Ramkumar, G.V. Anand and Ali N. Akansu*

New Jersey Center for Multimedia Research
Department of Electrical and Computer Engineering
New Jersey Institute of Technology
University Heights, Newark, NJ 07102-1982

* Dept of Electrical Communication Engineering
Indian Institute of Science
Bangalore, India 560012.

ABSTRACT

The concept of cyclic filter banks was introduced independently in Ref. [1] and [2], though in [1], a different nomenclature was used. "Cyclic filter banks" is a superset of the traditional non-cyclic filter banks, and readily suggest implementation of subband filtering through DFT. In this paper we show that the DFT implementation is also advantageous for the conventional FIR filter banks. The discussions in this paper concentrate on the more frequently used 2-band decompositions. However, extensions to M bands is not difficult.

1. INTRODUCTION

A concise theory of cyclic filter banks was presented in Ref [2]. In the traditional M-band, FIR perfect reconstruction (PR) filter banks, all the M filters are orthogonal to linear shifts by M. In cyclic filter banks on the other hand, the filters are orthogonal to cyclic shifts. As filters orthogonal to linear shifts are also orthogonal to cyclic shifts, cyclic filter banks have more degrees of freedom than the traditional FIR PR filter banks. Cyclic filter banks are thus a superset of the traditional non-cyclic filter banks (though it is a misnomer to call the traditional filter banks which *satisfy* the properties of cyclic filter banks *non-cyclic* it does serve to separate them from *strictly cyclic* filter banks.) One should also note that a strictly cyclic filter bank satisfying PR conditions for a length N, *will not* in general, satisfy the conditions for some other length $N1 \neq N$. On the other hand, conventional FIR PR-filter bank of length L will satisfy cyclic PR conditions if zero-padded to any length $N \geq L$.

The increased degrees of freedom for cyclic filter banks as opposed to non-cyclic filter banks promise more optimal filters. However, traditional optimization techniques (which are based on factorization of para-unitary systems [3]) used for deriving non-cyclic filter banks with desired characteristics can not be used for deriving strictly cyclic filter banks. This is due to the fact that strictly cyclic para-unitary systems can not be factorized like the non cyclic para-unitary systems [2].

It has been suggested in Ref. [4] that subband filtering can be implemented by cyclic filtering. In this paper we take a closer look at the algorithm for cyclic implementation of subband filtering. We show that cyclic implementation of subband filtering is much faster than one would expect at first glance. This is especially true for recursive filtering and non-separable 2-D filtering. The main purpose of this paper is to derive from first principles, an efficient implementation of cyclic filtering (especially for real data and real filters) and compare computational complexities of traditional FIR and cyclic implementations (in the traditional implementation, for a data length of N, and filter length L, each of the N subband coefficients is obtained as the inner-product of L neighboring data points with L filter coefficients). Obviously, since cyclic subband filters of length N are a superset of FIR subband filters of any length $L \leq N$ (and zero-padded to length N), the implementation proposed in this paper can also be used for FIR subband filters.

Let $\mathrm{x} \in \Re^N$ be the data vector. Let $\mathrm{h}, \mathrm{g} \in \Re^N$ be subband filters of support less than or equal to N. When the size of the data is finite, as in this case, then the two band subband decomposition of the data may be seen as a $N \times N$ block transform [5]. The orthonormal basis vectors of the transform matrix $\mathrm{T}_{N\times N}$ are h and other $N/2-1$ vectors obtained by alternate cyclic shifts of h, and g and $N/2-1$ vectors obtained from alternate cyclic shifts of g. The subband coefficients are the projections of the data vector x on the basis vectors. Therefore the $N/2$ subband coefficients corresponding to each of the filters h and g respectively are obtained as

$$x_h(m) = \sum_{n=0}^{N-1} x(n)h(n-2m), \; m = 0, \ldots, \frac{N}{2}-1, \qquad (1)$$

$$x_g(m) = \sum_{n=0}^{N-1} x(n)g(n-2m), \; m = 0, \ldots, \frac{N}{2}-1. \qquad (2)$$

Obviously, x can be obtained from x_h and x_g using the inverse transform matrix, which in this case is just the transpose of T.

In the next section we show how the filters h and g are derived. In Section 3 we show how the forward and inverse transforms can be implemented efficiently using the FFT algorithm.

2. CYCLIC 2-BAND FILTERBANKS

Let $\mathrm{h} \in \Re^N$ and $\mathrm{h} \leftrightarrow \mathrm{H}$, where $\leftrightarrow$ denotes a discrete Fourier transform (DFT) pair. Let

$$h_e(n) = h(2n), h_o(n) = h(2n+1),\ n = 0, \ldots, \frac{N}{2} - 1. \quad (3)$$

As h is orthogonal to alternate *cyclic* shifts,

$$\sum_{n=0}^{\frac{N}{2}-1} \{h_e(n)h_e(n-p) + h_o(n)h_o(n-p)\} = \delta(p). \quad (4)$$

Let $\mathrm{H}_e \leftrightarrow \mathrm{h}_e$ and $\mathrm{H}_o \leftrightarrow \mathrm{h}_o$. Taking the DFT of both sides of Eqn. (4),

$$\mathrm{H}_e.\mathrm{H}_e^* + \mathrm{H}_o.\mathrm{H}_o^* = [1\ 1\ \cdots\ 1] \in \Re^{\frac{N}{2}} \quad (5)$$

where $(, .,)$ stands for the Hadamard product (multiplication of corresponding elements) of two vectors. It can be easily shown that the l^{th} elements of H_e and H_o are given by

$$\begin{aligned} H_e(l) &= \sum_{n=0}^{\frac{N}{2}-1} h(2n) \exp\left(\frac{-j2\pi nl}{\frac{N}{2}}\right) = \frac{H(l) + H(l+\frac{N}{2})}{2} \\ H_o(l) &= \frac{1}{2} \exp\left(\frac{j2\pi l}{N}\right) \Big[H(l) - H(l+\frac{N}{2})\Big]. \end{aligned} \quad (6)$$

Substituting Eqn. (6) into Eqn. (5) and simplifying,

$$|H(l)|^2 + |H(l+\frac{N}{2})|^2 = 2 \text{ for } l = 0, \cdots, \frac{N}{2} - 1. \quad (7)$$

Equation (7) is a necessary and sufficient condition for the vector h to be orthogonal to all its alternate circular shifts. Note that in addition to the freedom in selecting the DFT magnitudes of H, there is complete freedom in the choice of their phases (except, of course if h has to be real, only $\frac{N}{2} - 1$ phase values are independent). Now $\frac{N}{2}$ orthonormal basis vectors can be obtained from h. We now want to obtain $\frac{N}{2}$ complementary basis vectors, to complete the basis for $\Re^N$. Let g be a vector which is also orthogonal to its alternate shifts. Then

$$|G(l)|^2 + |G(l+\frac{N}{2})|^2 = 2 \text{ for } l = 0, \cdots, \frac{N}{2} - 1. \quad (8)$$

Since we desire g and its alternate cyclic shifts to complement the basis vectors derived from h, g should further satisfy

$$\sum_{n=0}^{\frac{N}{2}-1} \{h_e(n)g_e(n-p) + h_o(n)g_o(n-p)\} = 0, \quad (9)$$

where, $g_e(n)$ and $g_o(n)$ are respectively the even and odd indexed elements of g. Taking the DFT of Eqn. (9),

$$H_e(k)G_e^*(k) + H_o(k)G_o^*(k) = 0\ \forall\ k. \quad (10)$$

Using Eqn. (6), and similar relations for $G_e(l)$ and $G_o(l)$, Eqn. (10) can be rewritten as

$$H(k)G^*(k) = -H(k+\frac{N}{2})G^*(k+\frac{N}{2}). \quad (11)$$

Equation (11) is satisfied if we choose

$$G(k) = H^*(k+\frac{N}{2}) \exp\left(\frac{j2\pi k}{N}\right) \exp(j\theta) \quad (12)$$

where θ is an arbitrary phase angle. Choosing $\theta = 0$, we get

$$g(n) = (-1)^{n-1} h(N-1-n). \quad (13)$$

3. FAST IMPLEMENTATION OF CYCLIC SUBBAND FILTERING

In this section, we will show that the cyclic subband decomposition and reconstruction can be implemented efficiently using the FFT algorithm.

3.1. Cyclic Decomposition or Forward Transform

Define

$$y_h(m) = \sum_{n=0}^{N-1} x(n)h(n-m),\ m = 0, \cdots, N-1 \quad (14)$$

and

$$y_g(m) = \sum_{n=0}^{N-1} x(n)g(n-m),\ m = 0, \cdots, N-1. \quad (15)$$

Let $\mathrm{Y}_h \leftrightarrow \mathrm{y}_h$ and $\mathrm{Y}_g \leftrightarrow \mathrm{y}_g$. Taking the DFT of Eqns. (14) and (15),

$$Y_h(k) = X(k)H^*(k), \text{ and } Y_g(k) = X(k)G^*(k). \quad (16)$$

In view of Eqn. (17), we can obtain the transform coefficients $x_h(m)$ and $x_g(m)$ by sub-sampling the IDFTs of Y_h and Y_g. Alternatively, from Eqns (1),(2), (14), and (15), we have

$$x_h(m) = y_h(2m);\ x_g(m) = y_g(2m). \quad (17)$$

Therefore,

$$\begin{aligned} x_h(m) &= y_h(2m) = \frac{1}{N} \sum_{k=0}^{N-1} Y_h(k) \exp\left(\frac{j4\pi mk}{N}\right) \\ &= \frac{1}{N} \sum_{k=0}^{\frac{N}{2}-1} Z_h(k) \exp\left(\frac{j4\pi mk}{N}\right), \end{aligned} \quad (18)$$

where

$$Z_h(k) = Y_h(k) + Y_h(k+\frac{N}{2}),\ k = 0, \ldots, \frac{N}{2} - 1. \quad (19)$$

Similarly,

$$x_g(m) = \frac{1}{N} \sum_{k=0}^{\frac{N}{2}-1} Z_g(k) \exp\left(\frac{j4\pi mk}{N}\right). \quad (20)$$

where

$$Z_g(k) = Y_g(k) + Y_g(k+\frac{N}{2}),\ k = 0, \ldots, \frac{N}{2} - 1. \quad (21)$$

Thus $x_h(m)$ and $x_g(m)$ can be determined by computing the $\frac{N}{2}$ -point IDFTs of Z_h and Z_g, instead of computing the N-point IDFTs of Y_h and Y_g and sub-sampling them.

The implementation of the forward transform of x thus consists of the following steps

1. Obtain the DFT X of x.
2. Compute the Hadamard products $\mathrm{Y}_h = \mathrm{X}.\mathrm{H}^*$ and $\mathrm{Y}_g = \mathrm{X}.\mathrm{G}^*$.
3. Split the N-vector Y_h into two $\frac{N}{2}$-vectors and add them to obtain the $\frac{N}{2}$-vector Z_h. Form the $\frac{N}{2}$-vector Z_g from the N-vector Y_g in a similar fashion.
4. Obtain x_h and x_g as the IDFTs of Z_h and Z_g respectively.

3.2. Reconstruction from Cyclic Subband Coefficients - Inverse Transform

Let X_h and X_g denote the periodic extensions of the $\frac{N}{2}$-point DFTs of x_h and x_g respectively, i.e.,

$$X_h(k) = \sum_{m=0}^{\frac{N}{2}-1} x_h(m) \exp\left(\frac{-j4\pi km}{N}\right),\ k = 0, \ldots, N-1, \quad (22)$$

$$X_g(k) = \sum_{m=0}^{\frac{N}{2}-1} x_g(m) \exp\left(\frac{-j4\pi km}{N}\right),\ k = 0, \ldots, N-1, \quad (23)$$

It can be shown that (see Appendix)

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1}[X_h(k)H(k) + X_g(k)G(k)] \exp\left(\frac{j2\pi nk}{N}\right) \quad (24)$$

The implementation of the inverse transform therefore, consists of the following steps:

1. Obtain the $\frac{N}{2}$ length DFTs of x_h and x_g.
2. Make periodic extensions of these DFTs to length N to obtain X_h and X_g.
3. Compute the Hadamard products $\mathrm{X}_h.\mathrm{H}$ and $\mathrm{X}_g.\mathrm{G}$.
4. Compute the IDFT of $\mathrm{X}_h.\mathrm{H} + \mathrm{X}_g.\mathrm{G}$ to obtain x.

3.3. Computational Complexity

The implementation of cyclic subband decomposition and reconstruction (for a signal size of N) both require the computation of 2 FFTs of length $\frac{N}{2}$, 1 FFT of length N, and 2 Hadamard products of complex N-vectors. Since these complex N-vectors are DFT of real N-vectors, each Hadamard product involves about $\frac{N}{2}$ complex multiplications. Each complex multiplication can be implemented using 3 real multiplications [6, 7]. The computation of the FFT of a real signal of length N (where N is a power of 2) involves $\frac{N}{2}(\log_2 N - 3)$ multiplications [6, 7]. FFT implementation of subband filtering would therefore need $\frac{N}{2}(\log_2 N - 3) + 3N + 2\frac{N}{4}(\log_2 \frac{N}{2} - 3) = N(\log_2 N - 0.5)$ multiplications. On the other hand, the traditional FIR filter implementation requires NL multiplications. The FFT implementation would therefore be computationally less expensive if $L \geq \log_2 N$.

It is more common however to have many levels of decomposition. The k^{th} level of decomposition results in 2^k bands. Note that (for the intermediate levels) we do not need to obtain the size $\frac{N}{2^k}$-IDFTs of level k and therefore the size $\frac{N}{2^k}$-DFTs at the beginning of level $k+1$ (in other words, steps 1 and 4 of the forward transform algorithm). For the last level however, we have to perform the size $\frac{N}{2^k}$-IDFTs. The intermediate levels will just need Hadamard products of the outputs of the previous levels with the filters of the present level (in the DFT domain). On the other hand, conventional implementation of subband filtering has the same computational complexity for each level. So while a k level conventional subband decomposition (decomposition into 2^k bands) will need kNL multiplications, it can be easily seen that a cyclic filtering implementation will need only $N(\log_2 N + (5k-6)/2)$ (1 FFT of size N and Hadamard product for level 1, only Hadamard products for levels 2 to k-1, Hadamard product and 2^k FFTs of size $\frac{N}{2^k}$ for the k^{th} level). As an example, if $N = 256$ and $k = 5$ (or 32 band decomposition of a vector of length 256), the conventional implementation would need 10240 multiplications (for $L = 8$), while cyclic filtering would need only 4480 multiplications for any $L \leq N$. As another example, let us consider separable 2-D subband decomposition of an image of size 512×512 into 1024 spatial frequency bands. The traditional FIR filtering (with $L = 8$) would need 20971520 multiplications. On the other hand, cyclic implementation would need only 9699328 multiplications.

Perhaps the most important application of cyclic filtering would be for 2-D subband decomposition with *non-separable* filters. If $\mathrm{E}, \mathrm{F}, \mathrm{G}, \mathrm{H} \in \Re^{N \times N}$ are mutually orthogonal basis matrices, all of which are also orthogonal to their alternate shifts (in horizontal and vertical directions), then, the forward transform coefficients of $\mathrm{X} \in \Re^{N \times N}$ can be obtained the same way as the 1-D case, except that the 1-D DFT is replaced by 2-D DFT. If Y_e is the Hadamard product of the 2-D DFTs of E and X, then, the subband coefficients $X_e(m,n)$ (which are also the even indexed IDFT coefficients of Y_e), can be obtained as the $\frac{N}{2} \times \frac{N}{2}$ point 2-D IDFT of

$$\begin{aligned} Z_e(k,l) &= Y_e(k,l) + Y_e(k, l+N/2) + Y_e(k+N/2, l) \\ &+ Y_e(k+N/2, l+N/2), \end{aligned} \quad (25)$$

where, $k, l = 0 \cdots \frac{N}{2} - 1$. In the same fashion, the other forward transform coefficients $X_f(m,n)$, $X_g(m,n)$ and $X_h(m,n)$ can be obtained.

The reconstruction or inverse transform is obtained by taking the $\frac{N}{2} \times \frac{N}{2}$ point 2-D IDFTs of $X_e(m,n)$, $X_f(m,n)$, $X_g(m,n)$ and $X_h(m,n)$ and cyclically extending them to a size of $N \times N$. The Hadamard products of the cyclic extensions with the 2-D DFTs of the corresponding four filters are added together and the inverse 2-D DFT is taken to obtain X. Note that the computational complexity does not depend on whether the subband filters are separable or not. So this would facilitate non-separable subband filtering with computational complexity comparable to or most often less than conventional *separable* subband filtering.

4. CONCLUSIONS

In this paper we have made an in-depth analysis of the computational complexities of the traditional (FIR) implementation and cyclic implementations for subband filtering. We have shown, that cyclic filtering is definitely preferable for longer filters. It is especially advantageous to use cyclic implementations for recursive filtering, and when non-separable two dimensional filters are used.

5. APPENDIX

5.1. Proof of Eqn. (24)

$$\begin{aligned} x(n) &= \frac{1}{N}\sum_{k=0}^{N-1}[X_h(k)H(k) + X_g(k)G(k)] \exp\left(\frac{j2\pi nk}{N}\right) \\ &= T_1(n) + T_2(n). \end{aligned} \quad (26)$$

Consider the first term, $T_1(n)$ of (26),

$$T_1(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_h(k)H(k) \exp\left(\frac{j2\pi nk}{N}\right) \quad (27)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=0}^{\frac{N}{2}-1} x_h(m)\exp\left(\frac{-j4\pi mk}{N}\right)$$
$$H(k)\exp\left(\frac{j2\pi nk}{N}\right) \quad (28)$$

As $x_h(m) = y_h(2m)$, and $\mathrm{y}_h \leftrightarrow \mathrm{Y}_h$, we have

$$x_h(m) = \frac{1}{N}\sum_{l=0}^{N-1} Y_h(l)\exp\left(\frac{j4\pi ml}{N}\right) \quad (29)$$

Substituting for $x_h(m)$ from Eqn. (29) into Eqn. (28), we obtain

$$\begin{aligned} T_1(n) &= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{m=0}^{\frac{N}{2}-1}\sum_{l=0}^{N-1} Y_h(l)H(k) \\ &\quad \exp\left(\frac{j2\pi[2ml+nk-2mk]}{N}\right) \\ &= \frac{1}{N}\sum_{l=0}^{N-1}\sum_{m=0}^{\frac{N}{2}-1}\left\{\frac{1}{N}\sum_{k=0}^{N-1} H(k)\right. \\ &\quad \left.\exp\left(\frac{j2\pi k(n-2m)}{N}\right)\right\} Y_h(l)\exp\left(\frac{j4\pi ml}{N}\right) \\ &= \frac{1}{N}\sum_{l=0}^{N-1} Y_h(l)\sum_{m=0}^{\frac{N}{2}-1} h(n-2m)\exp\left(\frac{j4\pi ml}{N}\right). \end{aligned}$$

For even n, i.e. $n = 2q$, we have $h(n-2m) = h_e(q-m)$ (see Eqn. (3)). Therefore,

$$\begin{aligned} T_1(2q) &= \frac{1}{N}\sum_{l=0}^{N-1}\left\{\sum_{m=0}^{\frac{N}{2}-1} h_e(q-m)\exp\left(\frac{j4\pi ml}{N}\right)\right\} Y_h(l) \\ &= \frac{1}{N}\sum_{l=0}^{N-1}\left\{\sum_{p=q}^{q+1-\frac{N}{2}} h_e(p)\exp\left(\frac{-j4\pi lp}{N}\right)\right\} \\ &\quad Y_h(l)\exp\left(\frac{j4\pi lq}{N}\right). \\ &= \frac{1}{N}\sum_{l=0}^{N-1} H_e(l)Y_h(l)\exp\left(\frac{j4\pi lq}{N}\right). \end{aligned} \quad (30)$$

Substituting for $H_e(l)$ from Eqn. (6) into Eqn. (30),

$$\begin{aligned} T_1(n) &= \frac{1}{N}\sum_{l=0}^{N-1}\frac{1}{2}\left[H(l)+H(l+\frac{N}{2})\right]Y_h(l) \\ &\quad \exp\left(\frac{j2\pi ln}{N}\right) \text{ for even } n. \end{aligned} \quad (31)$$

Similarly it can be easily shown that

$$\begin{aligned} T_1(n) &= \frac{1}{N}\sum_{l=0}^{N-1}\frac{1}{2}\left[H(l)-H(l+\frac{N}{2})\right]Y_h(l) \\ &\quad \exp\left(\frac{j2\pi ln}{N}\right) \text{ for odd } n. \end{aligned} \quad (32)$$

Similar expressions can be derived for $T_2(n)$ to obtain

$$T_2(n) = \begin{cases} \frac{1}{N}\sum_{l=0}^{N-1}\frac{1}{2}\left[G(l)+G(l+\frac{N}{2})\right]Y_g(l)\exp\left(\frac{j2\pi ln}{N}\right) \\ \frac{1}{N}\sum_{l=0}^{N-1}\frac{1}{2}\left[G(l)-G(l+\frac{N}{2})\right]Y_g(l)\exp\left(\frac{j2\pi ln}{N}\right) \end{cases}$$

for even and odd n respectively.

In view of Eqns. (12) and (7)),

$$H^*(l)H(l+\frac{N}{2}) + G^*(l)G(l+\frac{N}{2}) = 0. \quad (33)$$

$$|H(l)|^2 + |G(l)|^2 = 2. \quad (34)$$

Combining Eqn. (16), viz.,

$$Y_h(k) = X(k)H^*(k), \text{ and } Y_g(k) = X(k)G^*(k),$$

with the equations for $T_1(n)$ and $T_2(n)$, and using Eqns. (33) and (34),

$$T_1(n) + T_2(n) = \frac{1}{N}\sum_{l=0}^{N-1} X(l)\exp\left(\frac{j2\pi ln}{N}\right) = x(n). \quad (35)$$

6. REFERENCES

[1] M. Ramkumar, 'A Class of Feature Highlighting Transforms', Chapter 5 in 'Some New Methods for Improved Fractal Image Compression', Master's Thesis, Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore, India, May 1996. (Also available from http://www-ec.njit.edu/ mxr0096).

[2] P.P.Vaidyanathan and Ahmet Kirac, 'Theory of Cyclic Filterbanks', Proceedings of ICASSP-97, Munich, Germany, Vol **III**, pp 2449-2452, April 1997.

[3] P.P.Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice Hall P T R, 1993.

[4] M J T Smith, S L Eddins, 'Analysis/ Synthesis Techniques for Subband Image Coding', IEEE Transaction on ASSP, Vol **38**(8), pp 1446-1456, August 1990.

[5] A. N. Akansu, R. A. Haddad, *Multiresolution Signal Decomposition: Transforms, Subbands and Wavelets*, Academic Press Inc. 1992.

[6] P.Duhamel, 'Algorithms Meeting the Lower Bounds of the Multiplicative Complexity of Length -2^n DFTs and Their Connection with Practical Applications,' *IEEE Trans. on ASSP*, **ASSP-38**, pp 1504-1511, Sep 1990.

[7] H.S.Malvar, *Signal Processing with Lapped Transforms*, Artech House Inc., Norwood, 1992.