Exercise 4.1.1

Prove the following languages are not regular:

(d) $L = \{0^n 1^m 0^n \mid n, m \in \mathbb{N}\}$.

Suppose that this language is regular. Then the pumping lemma says that there is an n so that for all strings w of length $|w| \geq n$, there is a decomposition $w = xyz$ with $|xy| < n$ so that $xy^p z \in L$ for all $p \geq 0$. Consider the string $0^n 1^m 0^n$. If $|xy| \leq n$ then $xy = 0^k$ for some $k \leq n$. Then $y = 0^j$ for some $j \leq k$. Then $xyz = 0^{k-j} 0^k z$ and $xy^2 z$ clearly has more zeros in the first part of the string than in the last part of the string. So $xy^2 z \ni L$. $\rightarrow\leftarrow$ Thus L is not regular.

(e) $L = \{0^n 1^m \mid n \leq m\}$.

Suppose this language is regular. Then the pumping lemma says that there is an n so that for all strings w with $|w| \geq n$ there is a decomposition $w = xyz$ so that $|xy| < n$, $y \neq \varepsilon$, and $xy^p z \in L$ for all $p \geq 0$. Consider the string $w = 0^n 1^m$. Then $xy = 0^k$ for some $k \leq n$ and $y = 0^j$ for some $j \leq k$. Then according to the pumping lemma, $xy^p z \in L$. But for large p, there will me more zeros in the string than ones. Thus those longer strings are not in L. $\rightarrow\leftarrow$ Thus L is not regular.

(f) $L = \{0^n 1^{2n} \mid n \in \mathbb{N}\}$.

Suppose this language is regular. Then the pumping lemma says that there is an n so that for all strings w with $|w| \geq n$ there is a decomposition $w = xyz$ so that $|xy| < n$, $y \neq \varepsilon$, and $xy^p z \in L$ for all $p \geq 0$. Consider the string $w = 0^n 1^{2n}$. Then $xy = 0^k$ for some $k \leq n$ and $y = 0^j$ for some $j \leq k$. Then according to the pumping lemma, $xy^p z \in L$. But for large p, there will me more zeros in the string than ones, nevermind exactly half the number of ones. Thus those longer strings are not in L. $\rightarrow\leftarrow$ Thus L is not regular.

Exercise 4.1.2

Prove that the following are not regular:

(c) $L = \{0^{2^k} \mid k \geq 0\}$

Suppose that L is regular. Then there is an n so that for all strings w of length greater than n, $w = xyz$, $|xy| \leq n$, $|y| > 0$, and $xy^p z \in L$ for all $p \geq 0$. Then $xy^2 z$ does not have length that is a power of two. So $xy^2 z \ni L$. $\rightarrow\leftarrow$ Thus L is not regular.

(d) L is the set of strings of zeros and ones whose length is a perfect square.

Suppose that L is regular. Then by the pumping lemma, there is an n so that for all strings w of length greater than n, $w = xyz$ so that $|xy| \leq n$, $|y| > 0$, and $xy^p z \in L$ for all $p \geq 0$. Then suppose that N is the smallest perfect square greater than or equal to n. Then let $w \in L$ so that $|w| = N$. Then there is a decomposition $w = xyz$ with $|xy| \leq n$, $|y| > 0$ and $xy^p z \in L$ for all $p \geq 0$ according to the pumping lemma. So $xy^2 z \in L$, but $|xy^2 z|$ cannot be a perfect square (the next perfect square is $N^2 + 2N + 1$ and $xy^2 z$ cannot have length more than $N + n$, which is not a perfect square. So $xy^2 z \ni L$. $\rightarrow\leftarrow$ Thus L is not regular.

(e) The set of strings of zeros and ones that are of the form ww.

Suppose this language is regular. Then by the pumping lemma, there is an n so that for all strings w of length greater than n, $w = xyz$ so that $|xy \leq n$, $|y| > 0$, and $xy^p z$ is in this language for all $p \geq 0$.

Now, consider the string $0^n 1^n 0^n 1^n$. Clearly this is in the language, but any decomposition that satisfies the pumping lemma criteria would have y consisting only of zeros, which, when repeated, would not form a string in this language. So there is a string of appropriate length that does not satisfy the criteria in the pumping lemma. $\rightarrow\leftarrow$. Thus this is not a regular language.

(f) The set of strings of zeros and ones that are of the form ww^R, that is, some string followed by its reversal.

Suppose this language is regular. Then there is some n so that for all strings w of length $|w| \geq n$, there is a decomposition $w = xyz$ so that $|xy| \leq n$, $|y| > 0$, and xy^pz is also in the language for all $p \geq 0$.

Let $w = 0^n1$. Then $|w| \geq n$ so there is a decomposition $w = xyz$ that satisfies the criteria in the pumping lemma. Clearly y must consist of one or more zeros, and "pumping" y does not produce strings that are in the language. There would be more zeros in the beginning of the string than in the end for $p > 1$ and fewer for $p = 0$. $\rightarrow\leftarrow$ Thus this language is not regular.

(g) The set of strings of zeros and ones of the form $w\overline{w}$ where $\overline{w}$ is the ones-complement of w.

Suppose that this language is regular. Then the pumping lemma says that there is an n so that for all strings w with length $|w| \geq n$, there is a decomposition $w = xyz$ so that $|xy| \leq n$, $|y| > 0$, and xy^pz is also in this language for all $p \geq 0$.

Let $w = 0^n1^n$. Clearly, the second half of this string is the ones-complement of the first half. Also, any decomposition of w that satisfies the pumping lemma will have y consisting of only zeros. Thus, when $p > 1$, the second half of the string will begin with zeros but the first half will not end with ones. Also, for $p = 0$, the first half ends with 1 but the second half does not begin with 0 (in fact, the second half consists only of ones). Thus xy^pz is not in this language for any $p \neq 1$. $\rightarrow\leftarrow$ Thus this is not a regular language.

(h) The set of strings of the form $w1^n$ where $|w| = n$.

Suppose that this language is regular. Then let n be such that the pumping lemma is satisfied. Consider the string 0^n1^n. Then y must contain one or more zeros. But by the pumping lemma, $xy^2z = 0^{n+k}1^n, k > 0$ must also be in the language. However, there is fewer 1's than 0's in this string, it cannot be in the language.$\rightarrow\leftarrow$ Thus this language is not regular.

Exercise 4.1.4

When we try to apply the pumping lemma to regular languages, the "adversary" wins and we cannot complete the proof. Show what goes wrong when we choose L to be one of the following languages:

(a) The empty set.

For any n, there are no strings of length greater than n, so this language vacuously satisfies the pumping lemma.

(b) $\{00, 11\}$

Let n be greater than 2. Then, as above, there are no strings of length greater than n, so the pumping lemma vacuously holds.

(c) $(00 + 11)^*$

For any n greater than 1, we can find a y of length 2 so that $y = 00$ or $y = 11$. Then we can repeat y as often as we like and we are simply choosing the 00 or 11 option repeatedly.

(d) 01^*0^*1

Choose n greater than 2. Then to get a string of length greater than n, we use either the "1^*" or "0^*" or both to make the string that long. Then put $x = 0$, put y equal to the second character in the string, and z the rest of the string. Then $|xy| < n$, $|y| > 0$, and xy^*z is clearly in this language.

For the last two languages, the adversary wins because the pumping lemma says that for any string there is a way to decompose it appropriately. That decomposition may be different for different strings and that's ok. The pumping lemma doesn't say that *all* strings must split into x, y, and z at the same points, nor that any particular decomposition would work, only that there exists such a decomposition. For non-regular (irregular?) languages, *no* decomposition will work.